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Heat transfer analysis of the steady flow of a fourth grade fluid

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Abstract

This paper deals with the influence of heat transfer on the flow of a fourth grade fluid past a porous plate. The heat transfer analysis has been carried out for the prescribed temperature. The series solution is first developed using homotopy analysis method (HAM) and then analyzed for its convergence. The obtained velocity profile is compared with the existing exact solution of the same flow problem for a second grade fluid. It is found that HAM results are in excellent agreement with the exact solution. Finally, the velocity and temperature profiles are plotted and discussed. It is noted from the solution series that the results in the case of injection does not exist.

Keywords: Porous plate; Fourth grade fluid; Heat transfer analysis; HAM solution

1. Introduction

Heat transfer from the wall is very important in industries processing molten plastics, polymers, food stuff or slurries. The need to understand such mechanism has led to a large number of experimental and theoretical studies. Considerable attention has been directed towards the analysis and understanding of such problems characterized by highly non-linear differential equations. Much efforts have been made for the Newtonian fluids. But non-Newtonian fluids having extensive applications in industry and technology have not been given proper attention. The dearth of studies for non-Newtonian flows with heat transfer, we believe, due partly to the inherently complex flow geometry combined with the difficulty to represent accurately the rheological behavior of non-Newtonian materials. Such flows with heat transfer have several applications in the environmental and industrial problems such as drying processes, heat pipe technologies, aquifers overlying salt formations or geothermal reservoirs and many others in geophysical fluid dynamics.

In view of the above motivation, the aim of the current article is to analyze the effect of heat transfer on the flow of a non-Newtonian fluid. The non-Newtonian fluid obeys the fourth order fluid model. The governing equations are considered for the flow past a porous plate. The plate and main free stream have different temperatures. Such analysis of heat transfer analysis in boundary layer flows is of great importance in many engineering applications. Such applications include the design of thrust bearings and radial diffusers, transpiration cooling, drag reduction, thermal recovery of oil etc.

The present article is organized as follows. In the next section, the equations describing the flow and heat transfer are presented. The third section includes the analytical solutions for the velocity and temperature using a powerful, recently developed technique namely the homotopy analysis method (HAM) by Liao [1,2]. This technique has already been used for the solution of various problems [3–26]. Recently, Sajid et al. [27] studied the flow of a fourth grade fluid past a porous plate. In this study the heat transfer is not considered. Here the heat transfer analysis is considered for the same problem. However, a better choice of the auxiliary linear operator for the velocity is made. It is noted that now an excellent agreement is achieved for the existing results of velocity in a second grade fluid [28]. The fourth section describes the convergence of the HAM solutions. Fifth section includes the results and discussion. The last section consists of concluding remarks.

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2. Statement of the problem

This paper is in continuation to our earlier work [27]. Here the steady flow and heat transfer analysis of a fourth grade fluid past a porous plate is considered. The x-axis is parallel to the plate and the y-axis is normal to it and the velocity field to depend only on y. The equations which govern the flow and heat transfer analysis are the incompressibility condition

$$\operatorname{div}\mathbf{V}=0 \tag{1}$$

the momentum equation

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}=\frac{1}{\rho}\operatorname{div}\mathbf{T} \tag{2}$$

and the energy equation

$$\rho c_p\frac{\mathrm{d}\theta}{\mathrm{d}t}=k\nabla^2\theta+\mathbf{T}\cdot\mathbf{L} \tag{3}$$

where ρ is the fluid density, c_p is the specific heat, k is the thermal conductivity, $\mathbf{T}$ is the velocity, θ is the temperature and $\mathbf{T}$ is the Cauchy stress tensor. For the fourth grade fluid we have

$$\mathbf{T}=-p\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\mathbf{S}_1+\mathbf{S}_2 \tag{4}$$

$$\mathbf{S}_1=\beta_1\mathbf{A}_3+\beta_2(\mathbf{A}_2\mathbf{A}_1+\mathbf{A}_1\mathbf{A}_2)+\beta_3(\operatorname{tr}\mathbf{A}_1^2)\mathbf{A}_1 \tag{5}$$

$$\begin{aligned}\mathbf{S}_2={}&\gamma_1\mathbf{A}_4+\gamma_2(\mathbf{A}_3\mathbf{A}_1+\mathbf{A}_1\mathbf{A}_3)+\gamma_3\mathbf{A}_2^2\\&+\gamma_4\big(\mathbf{A}_2\mathbf{A}_1^2+\mathbf{A}_1^2\mathbf{A}_2\big)+\gamma_5(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_2\\&+\gamma_6(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_1^2+\big(\gamma_7\operatorname{tr}\mathbf{A}_3+\gamma_8\operatorname{tr}(\mathbf{A}_2\mathbf{A}_1)\big)\mathbf{A}_1\end{aligned} \tag{6}$$

where p is the hydrostatic pressure, $\mathbf{I}$ is the identity tensor, μ is the coefficient of viscosity and α_i $(i=1,2)$, β_i $(i=1–3)$ and γ_i $(i=1–8)$ are the material constants. It should be noted that for Navier–Stokes fluid, α_i $(i=1,2)=\beta_i$ $(i=1–3)=\gamma_i$ $(i=1–8)=0$. When α_i $(i=1,2)\neq 0$ and β_i $(i=1–3)=\gamma_i$ $(i=1–8)=0$ then we get the second order fluid model. Furthermore, when α_i $(i=1,2)\neq 0$ and β_i $(i=1–3)\neq 0$, γ_i $(i=1–8)=0$, one obtains the third order fluid model. The kinematical tensors $\mathbf{A}_1$ to $\mathbf{A}_4$ are the Rivlin–Ericksen tensors given by

$$\mathbf{A}_1=\mathbf{L}+\mathbf{L}^\top \tag{7}$$

$$\mathbf{A}_n=\frac{\mathrm{d}\mathbf{A}_{n-1}}{\mathrm{d}t}+\mathbf{A}_{n-1}\mathbf{L}+\mathbf{L}^\top\mathbf{A}_{n-1}\quad(n>1) \tag{8}$$

$$\mathbf{L}=\nabla\mathbf{V} \tag{9}$$

Under the consideration of flow, it follows from Eq. (1) that for uniformly porous plate

$$v=-V_0 \tag{10}$$

where $V_0>0$ is the suction velocity and $V_0<0$ corresponds to the injection velocity.

Since the velocity and temperature depends only on y, therefore Eqs. (2) and (3) takes the form

$$-\rho V_0\frac{\mathrm{d}u}{\mathrm{d}y}=\frac{\mathrm{d}}{\mathrm{d}y}T_{xy} \tag{11}$$

$$-\rho c_p V_0\frac{\mathrm{d}\theta}{\mathrm{d}y}=k\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2}+T_{xy}\frac{\mathrm{d}u}{\mathrm{d}y} \tag{12}$$

$u(y)$ is the x-component of velocity and the shear stress T_{xy} is [27]

$$\begin{aligned}T_{xy}={}&\mu\frac{\mathrm{d}u}{\mathrm{d}y}-\alpha_1V_0\frac{\mathrm{d}^2u}{\mathrm{d}y^2}+\beta_1V_0^2\frac{\mathrm{d}^3u}{\mathrm{d}y^3}\\&+2(\beta_2+\beta_3)\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3-\gamma_1V_0^3\frac{\mathrm{d}^4u}{\mathrm{d}y^4}\\&-(6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)V_0\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\end{aligned} \tag{13}$$

The boundary conditions for the problem under consideration are

$$\begin{aligned}&u(0)=0,\quad\text{and}\\&u(y)\to U_0,\quad\frac{\mathrm{d}^nu}{\mathrm{d}y^n}\to 0\quad\text{as } y\to\infty \text{ for } n=1,2,3\\&\theta(0)=\theta_0,\quad\text{and}\quad\theta(y)\to\theta_\infty \text{ as } y\to\infty\end{aligned} \tag{14}$$

where U_0 is the constant velocity of the free stream, θ_0 is the wall temperature and θ_∞ is the temperature of the free stream. Defining

$$\begin{aligned}&\bar u=\frac{u}{U_0},\qquad\bar\theta=\frac{\theta-\theta_\infty}{\theta_0-\theta_\infty},\qquad\bar y=\frac{U_0y}{v}\\&\overline{V}_0=\frac{V_0}{U_0},\qquad\bar\alpha_1=\frac{\alpha_1U_0^2}{\rho v^2},\qquad\bar\beta_1=\frac{\beta_1U_0^4}{\rho v^3}\\&\bar\gamma_1=\frac{\gamma_1U_0^6}{\rho v^4},\qquad\bar\beta=\frac{6(\beta_2+\beta_3)U_0^4}{\rho v^3}\\&\bar\gamma=2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)\frac{U_0^6}{\rho v^4}\end{aligned}$$

Eqs. (11) and (12) and boundary conditions (14) simplify to

$$\begin{aligned}&\frac{\mathrm{d}^2u}{\mathrm{d}y^2}+V_0\frac{\mathrm{d}u}{\mathrm{d}y}-\alpha_1V_0\frac{\mathrm{d}^3u}{\mathrm{d}y^3}+\beta_1V_0^2\frac{\mathrm{d}^4u}{\mathrm{d}y^4}\\&\quad-\gamma_1V_0^3\frac{\mathrm{d}^5u}{\mathrm{d}y^5}+\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\\&\quad-\gamma V_0\left\{2\frac{\mathrm{d}u}{\mathrm{d}y}\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)^2+\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\frac{\mathrm{d}3u}{\mathrm{d}y^3}\right\}=0\end{aligned} \tag{15}$$

$$\begin{aligned}&\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2}+P_r\Bigg[V_0\frac{\mathrm{d}\theta}{\partial y}\\&\quad+E_c\left\{\begin{matrix}\left(\frac{\mathrm{d}u}{\partial y}\right)^2-\alpha_1V_0\frac{\mathrm{d}^2u}{\partial y^2}\frac{\mathrm{d}u}{\partial y}+\beta_1V_0^2\frac{\mathrm{d}^3u}{\mathrm{d}y^3}\frac{\mathrm{d}u}{\mathrm{d}y}\\+\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^4-\gamma V_0\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3\frac{\mathrm{d}^2u}{\mathrm{d}y^2}-\gamma_1V_0^3\frac{\mathrm{d}^4u}{\mathrm{d}y^4}\frac{\mathrm{d}u}{\mathrm{d}y}\end{matrix}\right\}\Bigg]=0\end{aligned} \tag{16}$$

$$\begin{aligned}&u(0)=0,\quad\text{and}\\&u\to 1,\quad\frac{\mathrm{d}^nu}{\mathrm{d}y^n}\to 0\quad\text{as } y\to\infty \text{ for } n=1,2,3\\&\theta(0)=1,\quad\text{and}\quad\theta\to 0,\ \text{as } y\to\infty\end{aligned} \tag{17}$$

where $Pr=\mu c_p/k$, $E=U_0^2/c_p(\theta_0-\theta_\infty)$ are respectively the Prandtl and Eckert numbers and bars have been suppressed throughout.

3. HAM-solutions for velocity and temperature

3.1. Zeroth-order deformation problems

The velocity and temperature distributions $f(\eta)$ and $\theta(\eta)$ can be expressed by the set of base functions of the form

$$\left\{y^k \exp(-nV_0 y) \mid k \geqslant 0,\ n \geqslant 0\right\} \tag{18}$$

in the form of the following series

$$f(y) = a_{0,0}^0 + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{m,n}^k y^k \exp(-nV_0\eta)$$
$$\theta(\eta) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{m,n}^k y^k \exp(-nV_0 y) \tag{19}$$

in which $a_{m,n}^k$ and $b_{m,n}^k$ are the coefficients. Upon making use of the so-called *Rule of solution expressions* for $f(\eta)$ and $\theta(\eta)$ and Eqs. (15)–(17), the initial guesses $f_0(\eta)$, $\theta_0(\eta)$ and linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are

$$u_0(y) = 1 - \mathrm{e}^{-V_0 y}, \qquad \theta_0(y) = \mathrm{e}^{-V_0 y} \tag{20}$$
$$\mathcal{L}_1(u) = u'' + V_0 u', \qquad \mathcal{L}_2(\theta) = \theta'' - V_0^2\theta \tag{21}$$

which satisfy

$$\mathcal{L}_1\left[C_1 + C_2\mathrm{e}^{-V_0 y}\right] = 0, \qquad \mathcal{L}_2\left[C_3\mathrm{e}^{-V_0 y} + C_4\mathrm{e}^{V_0 y}\right] = 0 \tag{22}$$

where C_i: $i = 1, 2, 3, 4$ are arbitrary constants. Eqs. (15) and (16) show that the non-linear operators are

$$\begin{aligned}\mathcal{N}_1\left[\bar{u}(y,p)\right] = {} & \frac{\partial^2\bar{u}(y,p)}{\partial y^2} + V_0\frac{\partial\bar{u}(y,p)}{\partial y} - \alpha_1 V_0\frac{\partial^3\bar{u}(y,p)}{\partial y^3} \\ & + \beta_1 V_0^2\frac{\partial^4\bar{u}(y,p)}{\partial y^4} - \gamma_1 V_0^3\frac{\partial^5\bar{u}(y,p)}{\partial y^4} \\ & + \beta\left(\frac{\partial\bar{u}(y,p)}{\partial y}\right)^2\frac{\partial^2\bar{u}(y,p)}{\partial y^2} \\ & - \gamma V_0\left[\begin{array}{c}\left(\frac{\partial\bar{u}(y,p)}{\partial y}\right)^2\frac{\partial^3\bar{u}(y,p)}{\partial y^3} \\ +2\frac{\partial\bar{u}(y,p)}{\partial y}\left(\frac{\partial^2\bar{u}(y,p)}{\partial y^2}\right)^2\end{array}\right]\end{aligned} \tag{23}$$

$$\begin{aligned}\mathcal{N}_2\left[\bar{u}(y,p), \bar{\theta}(y,p)\right] = {} & \frac{\partial^2\bar{\theta}(y,p)}{\partial y^2} \\ & + P_r\left[\begin{array}{c}V_0\frac{\partial\bar{\theta}(y,p)}{\partial y} \\ +E\frac{\partial\bar{u}(y,p)}{\partial y}\left\{\begin{array}{c}\frac{\partial\bar{u}(y,p)}{\partial y} - \alpha_1 V_0\frac{\partial^2\bar{u}(y,p)}{\partial y^2} \\ +\beta_1 V_0^2\frac{\partial^3\bar{u}(y,p)}{\partial y^3} \\ -\gamma_1 V_0^3\frac{\partial\bar{u}^4(y,p)}{\partial y^4} \\ +\beta\left(\frac{\partial\bar{u}(y,p)}{\partial y}\right)^3 \\ -\gamma V_0\left(\frac{\partial\bar{u}(y,p)}{\partial y}\right)^2\frac{\partial^2\bar{u}(y,p)}{\partial y^2}\end{array}\right\}\end{array}\right]\end{aligned} \tag{24}$$

Denoting $\hbar$ and $\hbar_1$ as a non-zero auxiliary parameters, the zeroth order deformation problems are:

$$(1-p)\mathcal{L}_1\left[\bar{u}(y,p) - u_0(y)\right] = p\hbar\mathcal{N}_1\left[\bar{u}(y,p)\right] \tag{25}$$
$$(1-p)\mathcal{L}_2\left[\bar{\theta}(y,p) - \theta_0(y)\right] = p\hbar_1\mathcal{N}_2\left[\bar{u}(y,p), \bar{\theta}(y,p)\right] \tag{26}$$
$$\begin{aligned}&\bar{u}(0,p) = \bar{u}'(\infty,p) = \bar{u}''(\infty,p) = \bar{u}'''(\infty,p) = 0, \\ &\bar{u}(\infty,p) = 1, \qquad \bar{\theta}(0,p) = 1, \qquad \bar{\theta}(\infty,p) = 0\end{aligned} \tag{27}$$

in which $p \in [0,1]$ is the embedding parameter. When $p = 0$ and $p = 1$, we respectively have

$$\begin{aligned}&\bar{u}(y,0) = u_0(y), \qquad \bar{u}(y,1) = u(y) \\ &\bar{\theta}(y,0) = \theta_0(y), \qquad \bar{\theta}(y,1) = \theta(y)\end{aligned} \tag{28}$$

As p increases from 0 to 1, $\bar{u}(y,p), \bar{\theta}(y,p)$ varies $u_0(y), \theta_0(y)$ to the exact solutions $u(y)$, $\theta(y)$. By Taylor's theorem and Eq. (28) we can write

$$\bar{u}(y,p) = u_0(y) + \sum_{m=1}^{\infty} u_m(y)p^m$$
$$\bar{\theta}(y,p) = \theta_0(y) + \sum_{m=1}^{\infty} \theta_m(y)p^m \tag{29}$$

where

$$u_m(y) = \frac{1}{m!}\frac{\partial^m\bar{u}(y,p)}{\partial p^m}\bigg|_{p=0}$$
$$\theta_m(y) = \frac{1}{m!}\frac{\partial^m\bar{\theta}(y,p)}{\partial p^m}\bigg|_{p=0} \tag{30}$$

It is important to mention here that the two series given in Eq. (17) involve the auxiliary parameters $\hbar$ and $\hbar_1$ which determine the convergence of the series solutions. Assume that $\hbar$ and $\hbar_1$ are chosen in such a way that the series (17) are convergent at $p = 1$. Then due to Eq. (16) we have

$$u(y) = u_0(y) + \sum_{m=1}^{\infty} u_m(y)$$
$$\theta(y) = \theta_0(y) + \sum_{m=1}^{\infty} \theta_m(y) \tag{31}$$

3.2. mth-order deformation problems

Differentiating m times the zeroth order deformation equations (25) and (26) with respect to p, dividing by $m!$ and finally setting $p = 0$, the mth-order deformation problems are

$$\mathcal{L}_1\left[u_m(y) - \chi_m u_{m-1}(y)\right] = \hbar\mathcal{R}_m^1(y) \tag{32}$$
$$\mathcal{L}_2\left[\theta_m(y) - \chi_m\theta_{m-1}(y)\right] = \hbar_1\mathcal{R}_m^2(y) \tag{33}$$
$$\begin{aligned}u_m(0) &= u_m(\infty) = u_m'(\infty) = u_m''(\infty) = u_m'''(\infty) \\ &= \theta_m(0) = \theta_m(\infty) = 0\end{aligned} \tag{34}$$
$$\begin{aligned}\mathcal{R}_m^1(y) = {} & u_{m-1}''(y) + V_0 u_{m-1}'(y) - \alpha_1 V_0 u_{m-1}'''(y) \\ & + \beta_1 V_0^2 u_{m-1}^{\mathrm{iv}}(y) - \gamma_1 V_0^3 u_{m-1}^{v}(y) \\ & + \sum_{k=0}^{m-1} u_{m-1-k}'(y)\sum_{l=0}^{k}\left[\beta u_{k-l}'(y)u_l''(y)\right. \\ & \left. - \gamma V_0\left\{u_{k-l}'(y)u_l'''(y) + 2u_{k-l}''(y)u_l''(y)\right\}\right]\end{aligned} \tag{35}$$
$$\begin{aligned}\mathcal{R}_m^2(y) = {} & \theta_{m-1}''(y) \\ & + P_r\left[\begin{array}{c}V_0\theta_{m-1}'(y) + E\sum_{k=0}^{m-1} u_{m-1-k}'(y)\left\{\begin{array}{c}u_k'(y) - \alpha_1 V_0 u_k''(y) \\ +\beta_1 V_0^2 u_k'''(y) - \gamma_1 V_0^3\theta_k^{\mathrm{iv}}(y)\end{array}\right\} \\ +E\sum_{k=0}^{m-1} u_{m-1-k}'(y)\sum_{l=0}^{k} u_{k-l}'(y)\sum_{r=0}^{l}\left[\begin{array}{c}\beta u_{l-r}'(y)u_r'(y) \\ -\gamma V_0 u_{l-r}'(y)u_r''(y)\end{array}\right]\end{array}\right]\end{aligned} \tag{36}$$

where

$$\chi_m = \begin{cases} 0, & m \leqslant 1 \\ 1, & m > 1 \end{cases} \tag{37}$$

The system of non-linear equations (32)–(34) are solved up to first few order of approximations with the help of symbolic computation software MATHEMATICA. It is found that the solutions $u(y)$ and $\theta(y)$ are of the following form

$$u(y) = \sum_{m=0}^{\infty} u_m(y) = \lim_{M\to\infty}\left[a_{0,0}^0 + \sum_{n=1}^{2M+1} \mathrm{e}^{-nV_0 y}\left(\sum_{m=n-1}^{2M}\sum_{k=0}^{2m+1-n} a_{m,n}^k y^k\right)\right] \tag{38}$$

$$\theta(y) = \sum_{m=0}^{\infty} \theta_m(y) = \lim_{M\to\infty}\left[\sum_{n=1}^{2M+2} \mathrm{e}^{-nV_0 y}\left(\sum_{m=n-1}^{2M+1}\sum_{k=0}^{2m+2-n} \hat{a}_{m,n}^k y^k\right)\right] \tag{39}$$

where

$$a_{m,0}^k = \chi_m \chi_{2m+1-k} a_{m-1,0}^k, \quad 0 \leqslant k \leqslant 2m+1 \tag{40}$$

$$a_{m,1}^0 = \chi_m \chi_{2m} a_{m-1,1}^0 - \sum_{n=2}^{2m+1}\sum_{q=0}^{2m+1-n} \Gamma_{m,n}^q \mu_{n,0}^q \tag{41}$$

$$a_{m,1}^k = \chi_m \chi_{2m-k} a_{m-1,1}^k - \sum_{q=k-1}^{2m} \Gamma_{m,1}^q \mu_{1,k}^q, \quad 1 \leqslant k \leqslant 2m \tag{42}$$

$$a_{m,n}^k = \chi_m \chi_{2m+1-n-k} a_{m-1,n}^k - \sum_{q=k}^{2m+1-n} \Gamma_{m,n}^q \mu_{n,k}^q$$
$$2 \leqslant n \leqslant 2m+1,\ 0 \leqslant k \leqslant 2m+1-n \tag{43}$$

$$\hat{a}_{m,0}^k = \chi_m \chi_{2m+2-k} \hat{a}_{m-1,0}^k, \quad 0 \leqslant k \leqslant 2m+2 \tag{44}$$

$$\hat{a}_{m,1}^0 = \chi_m \chi_{2m+1} \hat{a}_{m-1,1}^0 - \sum_{n=2}^{2m+2}\sum_{q=0}^{2m+2-n} \hat{\Gamma}_{m,n}^q \hat{\mu}_{n,0}^q + \hat{\Gamma}_{m,0}^q \frac{1}{V_0^2} \tag{45}$$

$$\hat{a}_{m,1}^k = \chi_m \chi_{2m+1-k} \hat{a}_{m-1,1}^k - \sum_{q=k-1}^{2m+1} \hat{\Gamma}_{m,1}^q \hat{\mu}_{1,k}^q$$
$$1 \leqslant k \leqslant 2m+1 \tag{46}$$

$$\hat{a}_{m,n}^k = \chi_m \chi_{2m+2-n-k} \hat{a}_{m-1,n}^k - \sum_{q=k}^{2m+2-n} \hat{\Gamma}_{m,n}^q \hat{\mu}_{n,k}^q$$
$$2 \leqslant n \leqslant 2m+2,\ 0 \leqslant k \leqslant 2m+2-n \tag{47}$$

$$\mu_{1,k}^q = \frac{q!}{k! V_0^{q-k+2}}, \quad 0 \leqslant k \leqslant q+1,\ q \geqslant 0 \tag{48}$$

$$\mu_{n,k}^q = \sum_{p=0}^{q-k} \frac{q!}{k! V_0^{q-k+2}(n-1)^{q-k-p+1}(n)^{p+1}}$$
$$0 \leqslant k \leqslant q,\ q \geqslant 0, n \geqslant 2 \tag{49}$$

$$\hat{\mu}_{1,k}^q = \frac{q!}{k!(2V_0)^{q-k+2}}, \quad 0 \leqslant k \leqslant q+1,\ q \geqslant 0 \tag{50}$$

$$\hat{\mu}_{n,k}^q = \sum_{p=0}^{q-k} \frac{q!}{k! V_0^{q-k+2}(n-1)^{q-k-p+1}(n+1)^{p+1}}$$
$$0 \leqslant k \leqslant q,\ q \geqslant 0,\ n \geqslant 2 \tag{51}$$

$$\Gamma_{m,n}^q = \hbar\left[\begin{array}{c} \chi_{2m+1-n-q}\left\{\begin{array}{c} c_{m-1,n}^q + V_0 b_{m-1,n}^q - \alpha_1 V_0 d_{m-1,n}^q \\ +\beta_1 V_0^2 e_{m-1,n}^q - \gamma_1 V_0^3 f_{m-1,n}^q \end{array}\right\} \\ +\beta\delta_{m,n}^q - \gamma V_0 \Delta_{m,n}^q - 2\gamma V_0 \Lambda_{m,n}^q \end{array}\right] \tag{52}$$

$$\hat{\Gamma}_{m,n}^q = \hbar\left[\begin{array}{c} \chi_{2m+2-n-q}\left\{\begin{array}{c} \hat{c}_{m-1,n}^q \\ +P_r\left\{\begin{array}{c} V_0\hat{b}_{m-1,n}^q \\ +E\left\{\begin{array}{c} \delta_{m,n}^q - \alpha_1 V_0 \Delta_{m,n}^q \\ +\beta_1 V_0^2 \Lambda_{m,n}^q - \alpha_1 V_0^3 \Omega_{m,n}^q \end{array}\right\} \end{array}\right\} \end{array}\right\} \\ -P_r E_c\{\gamma V_0 \omega_{m,n}^q - \beta \Pi_{m,n}^q\} \end{array}\right] \tag{53}$$

$$\delta_{m,n}^q = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}}\ \sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}}\ \sum_{j=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}}\ \sum_{i=\max\{0,t-2k+2l-1+p-j\}}^{\min\{t,2l+1-j\}} c_{k,j}^i b_{k-l,p-j}^{t-i} b_{m-1-k,n-p}^{q-t} \tag{54}$$

$$\Delta_{m,n}^q = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}}\ \sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}}\ \sum_{j=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}}\ \sum_{i=\max\{0,t-2k+2l-1+p-j\}}^{\min\{t,2l+1-j\}} d_{l,j}^i b_{k-l,p-j}^{t-i} b_{m-1-k,n-p}^{q-t} \tag{55}$$

$$\Lambda_{m,n}^q = \sum_{k=0}^{m-1}\sum_{l=0}^{k}\sum_{p=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+2\}}\ \sum_{t=\max\{0,q-2m+2k+1+n-p\}}^{\min\{q,2k+2-p\}}\ \sum_{j=\max\{0,p-2k+2l-1\}}^{\min\{p,2l+1\}}\ \sum_{i=\max\{0,t-2k+2l-1+p-j\}}^{\min\{t,2l+1-j\}} c_{l,j}^i c_{k-l,p-j}^{t-i} b_{m-1-k,n-p}^{q-t} \tag{56}$$

$$\hat{\delta}_{m,n}^q = \sum_{k=0}^{m-1}\ \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}}\ \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} b_{m-l-k,n-j}^{q-i} b_{k,j}^i \tag{57}$$

$$\hat{\Delta}_{m,n}^q = \sum_{k=0}^{m-1}\ \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}}\ \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} c_{k,j}^i b_{m-1-k,n-j}^{q-i} \tag{58}$$

$$\hat{\Lambda}^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} d^i_{k,j} b^{q-i}_{m-1-k,n-j} \quad (59)$$

$$\Omega^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+1\}} \sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} e^i_{k,j} b^{q-i}_{m-1-k,n-j} \quad (60)$$

$$\omega^q_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{r=0}^{l} \sum_{p_1=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+3\}} \sum_{t_1=\max\{0,q-2m+2k+1+n-p_1\}}^{\min\{q,2k+3-p_1\}} \sum_{p=\max\{0,p_1-2k+2l-1\}}^{\min\{p_1,2l+2\}} \sum_{t=\max\{0,t_1,-2k+2l-1+p_1-p\}}^{\min\{t_1,2l+2-p\}} \sum_{j=\max\{0,p-2l+2r-1\}}^{\min=\{p,2r+1\}} \sum_{i=\max\{0,t-2l+2r+1+p-j\}}^{\min\{t,2l+2-p\}} c^i_{r,q-j} b^{t-i}_{l-r,p-j} b^{t_1-t}_{k-l,p_1-p} \times b^{q-t_1}_{m-1-k,n-p_1} \quad (61)$$

$$\Pi^q_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{r=0}^{l} \sum_{p_1=\max\{0,n-2m+2k+1\}}^{\min\{n,2k+3\}} \sum_{t_1=\max\{0,q-2m+2k+1+n-p_1\}}^{\min\{q,2k+3-p_1\}} \sum_{p=\max\{0,p_1-2k+2l-1\}}^{\min\{p_1,2l+2\}} \sum_{t=\max\{0,t_1,-2k+2l-1+p_1-p\}}^{\min\{t_1,2l+2-p\}} \sum_{j=\max\{0,p-2l+2r-1\}}^{\min=\{p,2r+1\}} \sum_{i=\max\{0,t-2l+2r+1+p-j\}}^{\min\{t,2l+2-p\}} b^i_{r,j} b^{t-i}_{l-r,p-j} b^{t_1-t}_{k-l,p_1-p} \times b^{q-t_1}_{m-1-k,n-p_1} \quad (62)$$

$$b^k_{m,n} = (k+1)a^{k+1}_{m,n} - nV_0 a^k_{m,n}$$

$$c^k_{m,n} = (k+1)b^{k+1}_{m,n} - nV_0 b^k_{m,n} \quad (63)$$

$$d^k_{m,n} = (k+1)c^{k+1}_{m,n} - nV_0 c^k_{m,n}$$

$$e^k_{m,n} = (k+1)d^{k+1}_{m,n} - nV_0 d^k_{m,n} \quad (64)$$

$$f^k_{m,n} = (k+1)d^{k+1}_{m,n} - nV_0 e^k_{m,n}$$

$$\hat{b}^k_{m,n} = (k+1)\hat{a}^{k+1}_{m,n} - nV_0 \hat{a}^k_{m,n} \quad (65)$$

$$\hat{c}^k_{m,n} = (k+1)\hat{b}^{k+1}_{m,n} - nV_0 \hat{b}^k_{m,n}$$

$$a^0_{0,0} = 1, \qquad a^0_{0,1} = -1, \qquad \hat{a}^0_{0,1} = 1 \quad (66)$$

The detail procedure of deriving the above recurrence relations is given in Ref. [4].

4. Convergence of the HAM solutions

The series in Eqs. (38) and (39) are the solutions for the velocity u and temperature θ if one guarantees the convergence of these series. As pointed out by Liao [2] the convergence and rate of approximation series strongly depends upon $\hbar$ and $\hbar_1$. The detail discussion on the role of the auxiliary parameters on the convergence region can be seen in Ref. [2] (pp. 31–33). Also a convergence theorem similar to one in Ref. [2] (pp. 18, 19) can be proved. In order to see the admissible range of the values of these auxiliary parameters, the $\hbar$-curves are drawn. In Fig. 1 the $\hbar$-curve for the velocity u is presented and it can be easily observed that the valid range for the values of $\hbar$ is $-1.5 < \hbar < -0.5$. Figs. 2–4 describe the influence of different physical parameters on the $\hbar$-curves. Fig. 2 demonstrates that by increasing the second grade parameter the interval of $\hbar$ shrinks towards -0.8. The similar behaviors are observed for the parameters β and V_0 (Figs. 3 and 4) but the variation in $\hbar$-curve is rapid for V_0 as compared to other parameters. It is worth pointing that even though the interval of the admissible values of $\hbar$ varies with the variation of parameters of the problem one can find a suitable choice of $\hbar$ that makes the solution series convergent. There is no change in the $\hbar$-curve for the variation of remaining material parameters like β_1, γ and γ_1. Fig. 4 shows that the valid range for the values of $\hbar_1$ is $-2.0 < \hbar_1 < -0.5$. To further strengthen the argument that the series solutions (26) and (27) are convergent we made a comparison between HAM solution and existing exact solution in

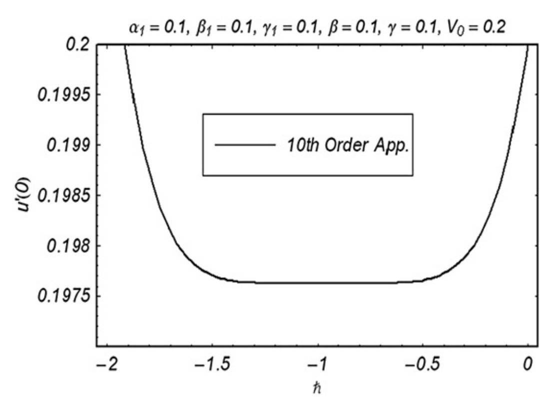

Fig. 1. $\hbar$-curve for the 10th order of approximation of u.

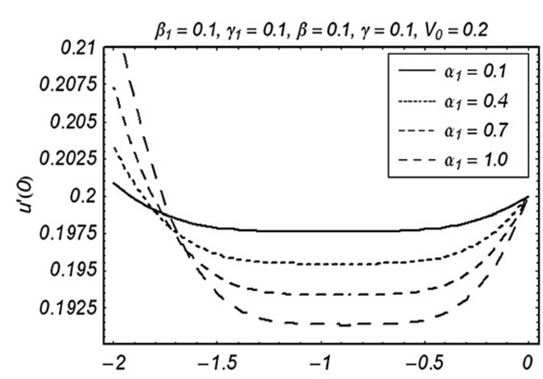

Fig. 2. Variation in $\hbar$-curve with increase in α_1.

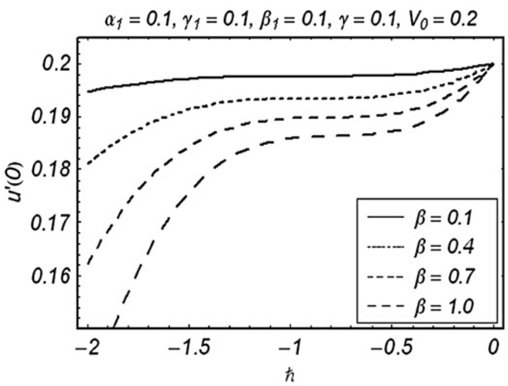

Fig. 3. Variation in $\hbar$-curve with increase in β.

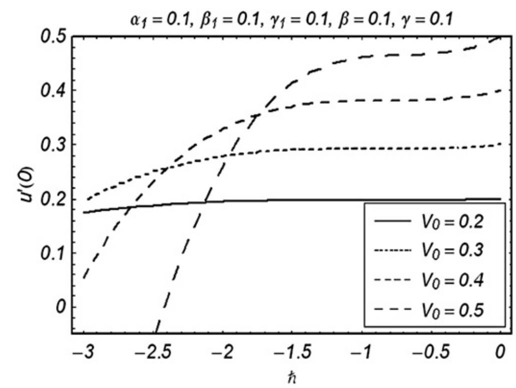

Fig. 4. Variation in $\hbar$-curve with increase in V_0.

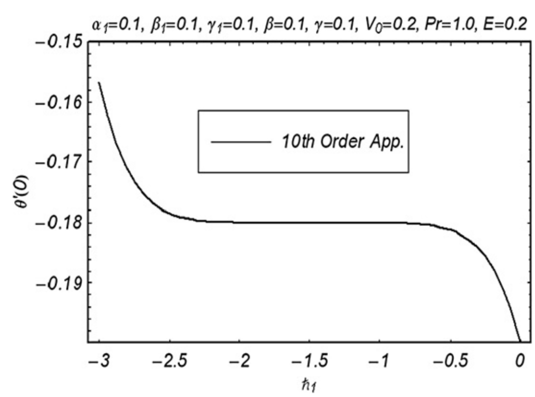

Fig. 5. $\hbar$-curve for the 10th order of approximation of θ.

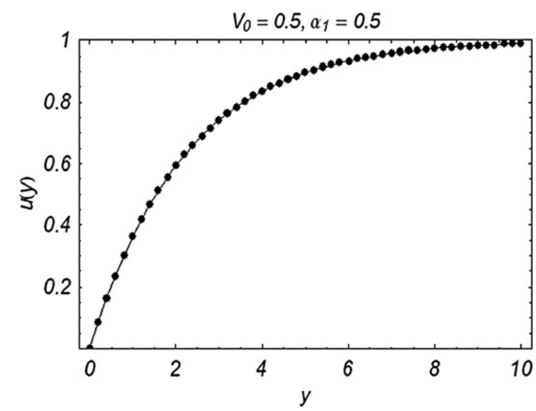

Fig. 6. Comparison between exact solution [28] and HAM solution.

the case of a second grade fluid [28]. Fig. 6 shows that the results of HAM are in excellent agreement with the exact solution [28].

5. Results and discussion

This section deals with the influence of the fluid parameters α_1, β, β_1, γ, γ_1 and the suction velocity V_0 on the velocity and temperature profiles. The effects of Prandtl number Pr and Eckert number E on the temperature profile is also included in this section. For this purpose, the Figs. 7–13 have been plotted.

Figs. 7 show that the velocity increases by an increase in the suction velocity V_0. The boundary layer thickness on the other hand decreases when suction velocity V_0 is increased. But suction velocity V_0 causes a decrease in both the temperature and thermal boundary layer thickness. Hence it is concluded that the suction on the plate can be used for controlling the boundary layer thickness. In fact for boundary layer control, there has been considerable interest in flows with suction or injection. It is apparent from physical consideration that suction and injection produce opposite effects on the boundary layer flows. It is already seen that when the suction velocity is strong enough the suction causes thickening of the boundary layer. Due to this reason by increasing the suction velocity boundary layer thickness decreases. But no steady asymptotic solution is possible for steady flow past a porous plate subjected to uniform injection. This is because of the fact that the injection causes a thickening of the boundary layer so that at a sufficiently large distance from the leading edge the boundary layer becomes so thick that it becomes turbulent and thus steady solution is not possible. It can be further seen from Figs. 8–12 that the effects of the material parameters of the fluid α_1, β, β_1, γ and γ_1 on the velocity and temperature profiles are quite opposite to that of suction velocity V_0, i.e. the velocity and temperature profiles decreases by increasing α_1, β, β_1, γ and γ_1. Moreover the velocity and thermal boundary layer thickness increases when α_1, β, β_1, γ and γ_1 are increased.

In order to describe the effects of Prandtl and Eckert numbers on the temperature Figs. 13 have been prepared. These figures elucidate that the temperature and thermal boundary layer thickness decreases by increasing the Prandtl number when Eckert number is fixed and behave in an opposite manner when we vary Eckert number keeping Prandtl number fixed. This is in accordance with the fact that effects of Prandtl and Eckert numbers are different.

6. Concluding remarks

Flow and heat transfer analysis of a fourth grade fluid over a porous plate is considered. The highly non-linear problems are analytically solved using homotopy analysis method. The convergence of the developed series solution is established and the recurrence formulas for finding the coefficients of the series are given. The effects of various parameters of interest on the ve-

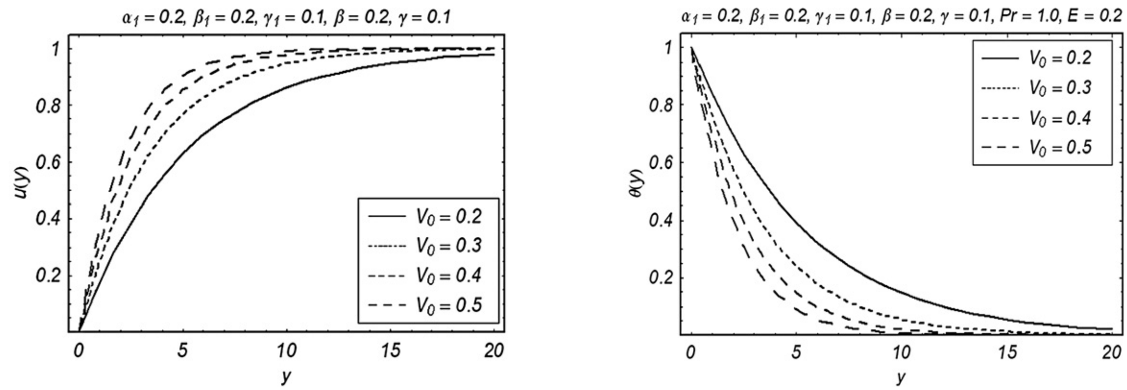

Fig. 7. Influence of suction parameter V_0 on velocity and temperature profiles.

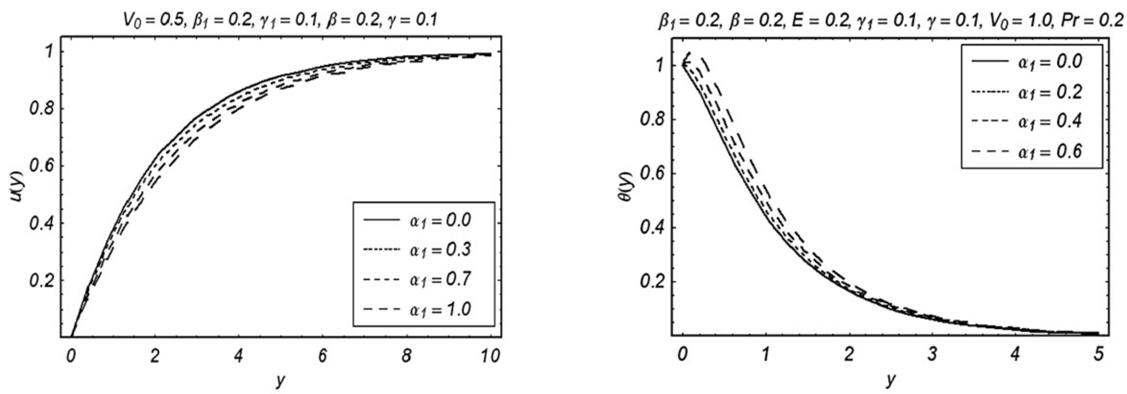

Fig. 8. Influence of material parameter α_1 on velocity and temperature profiles.

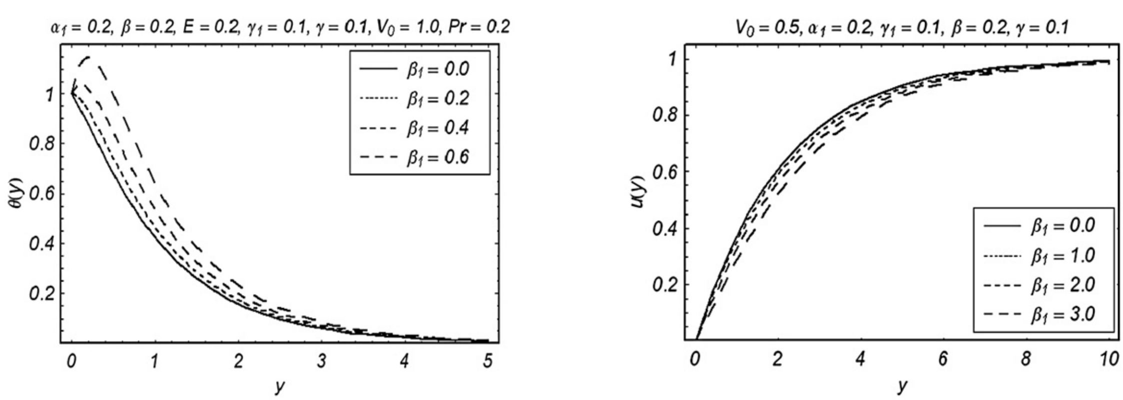

Fig. 9. Influence of material parameter β_1 on velocity and temperature profiles.

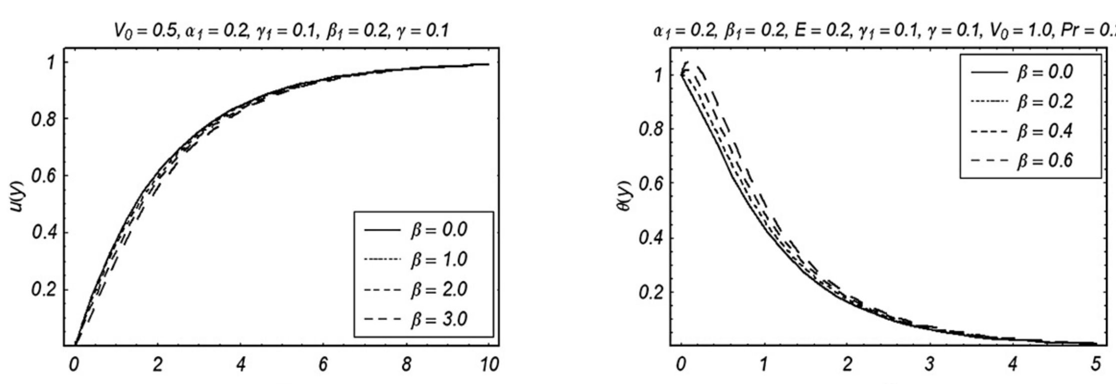

Fig. 10. Influence of material parameter β on velocity and temperature profiles.

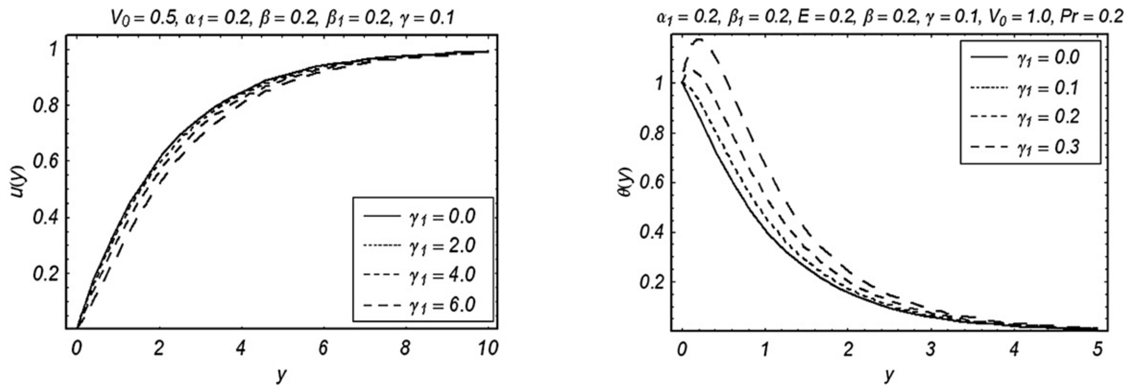

Fig. 11. Influence of material parameter γ_1 on velocity and temperature profiles.

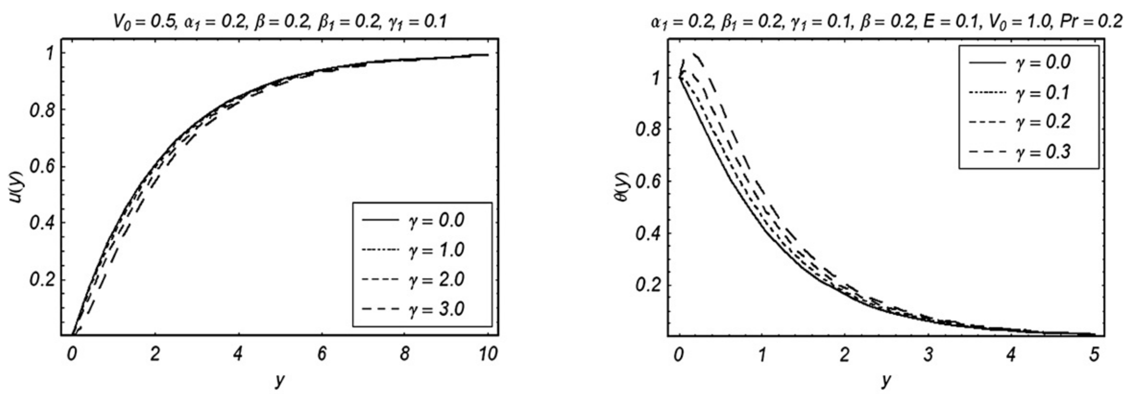

Fig. 12. Influence of material parameter γ on velocity and temperature profiles.

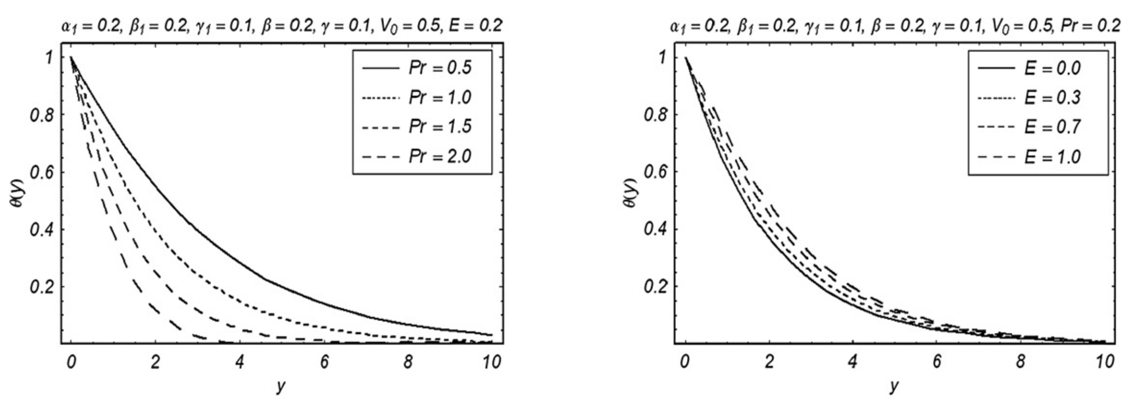

Fig. 13. Influence of Prandtl and Eckert numbers on temperature profile.

locity and temperature profiles are plotted and discussed. The comparison of the HAM results with the existing results in the literature is also presented. As expected it is found that the solution exists only for the suction case and is not possible for injection.

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